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DEVELOPMENT OF MATHEMATIC MODEL OF ELECTRON ENERGY TRANSPORT IN ELECTRIC PROPULSION DEVICES WITH CLOSED ELECTRON DRIFT

The object of research is a physical process of the electron energy transport in electric propulsion devices with closed electron drift. Such devices include ionization chambers of plasma-ion thrusters, Hall effect thrusters, helicon thrusters, and high-frequency plasma, ions and electrons sources where electron-electron collisions free path is small compared to the channel width. This means that the electron velocity distribution function cannot be considered as Maxwell.

The height of the potential barrier in the boundary bipolar layer and the average electrons energies removed from the plasma should be solved as a kinetic one. The presence of a potential barrier also means that only electrons with energies greater than the barrier height participate in mass and energy transfer. The classical representation, which represents the entire spectrum, is therefore inapplicable.

This means that the results of the research must be used on the object, and this will enable the object to improve, i. e. the electron energy transport must be described by expression obtained in this research.

The above problem is solved in this work using the tools of a compromise kinetic-fluid model considering the presence of isotropy factors of electrons velocity projections distribution. It has been shown that the removal of mass and energy from the plasma is carried out only by electrons in a narrow spectral band, about half the electron temperature. The equations of the zero and first angular moments of the distribution function are written with an approximate notation for the radial-azimuth component of the second angular moment as a second-rank tensor. It is shown that the ratio of the energy and mass flux densities in the volume is almost the same as that at the boundary with the bipolar layer, which allows to close the equations system of the mathematical model of processes in electric propulsion devices with a closed electron drift. The obtained results can be applied in the case of subsonic electron flow, which is typical for plasma of all types of electric propulsion devices.

Keywords: *closed electron drift, potential barrier, velocity distribution function, electrons mass and energy flux.*

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1. Introduction

Electric propulsion devices with closed electron drift (EPCD) include Hall effect thrusters (HET), plasma-ion thrusters with radial magnetic field (PIT-R) and RF devices: thrusters and sources of plasma, ions and electrons.

One of the main problems in modelling of processes inside the EPCD is the description of electrons energy transport across magnetic lines. The solution to this problem by means of gas dynamics involves writing a system of equations taking into account, among other things, heat conduction [1].

The paper [1] represents the classical description, which is based on the concept of the participation of the entire electron energy spectrum in the transport of their mass and energy and requires taking into account the influence of the magnetic field. In the above-mentioned devices, the movement of electrons is limited by potential barriers in the bipolar layer bordering the plasma [2], which actually means that only a small fraction, about 1%, of electrons in the energy region above the barrier participates in the mass and energy transfer.

The paper [3] represents the process of viscous transport of electron momentum in HET, which also contributes to the energy transfer, using also a classical expression that is not applicable in the rarefied plasma of EPCD.

The paper [4] represents an attempt to take into account the presence of a boundary layer, but using expressions based on the Maxwellian distribution and giving exaggerated values of the potential barrier height and the energy carried away by electrons from the plasma.

The paper [5] represents an attempt to correct Maxwellian expressions taking into account secondary electron-electron emission from the surface in the HET channel. However, the values of the barrier height and electron energy loss also differ from those calculated using Maxwellian equations in the external beam of the thruster, where secondary emission is absent.

Moreover, as shown in our work [6], the number of macroparticles in the Fluid-PIC model in works [3–5] corresponds to a relative standard deviation of about 10%, which even led to a violation of the mass conservation law in the calculation results in work [5].

Unfortunately, the analysis of works published for more than a quarter of a century, devoted to the Hybrid-PIC-Fluid model, indicates a lack of understanding by their authors of the origin and limits of applicability of the Maxwellian distribution, clearly shown in works [7, 8].

All this allows to state that it is advisable to conduct a research devoted to development of mathematic model of electron energy transport in electric propulsion devices with closed electron drift. The relevance of this research stems from the lack of consideration in existing mathematical models of the specific of electrons energy transport at presence of potential barrier inside boundary bipolar layer.

The object of research is a physical process of the electron energy transport in electric propulsion devices with closed electron drift.

The aim of research is to develop a mathematical model of processes taking the potential barrier influence on electrons dynamics as an integral part of holistic mathematical models of processes in various types of EPCD.

The objectives of this research are:

1. To propose the structure of a mathematical model including the expressions for the projections of the electron energy flux density, taking into account the influence of magnetic field with the use of compromise kinetic-fluid model [6].
2. To connect obtained results with boundary conditions for the electrons energy flux density.

2. Materials and Methods

2.1. Current state

The following scientific methods were used in the research:
– analysis method when studying existing models of processes in high-frequency cathodes;
– theoretical methods in formulating a mathematical model of electrons energy transport.

The input data for solving the problem were:
– fundamental equations of theoretical physics and plasma dynamics presented in the textbook [7] and the online lecture course [8];
– equations of the compromise kinetic-fluid model [6] developed for the first time with the participation of the author;
– the mathematical apparatus of the kinetic-fluid model [9], developed for the first time with the participation of the author, and made it possible to overcome the differences in symbolism in scalar-vector and tensor analysis and created the possibility of writing a universal equation (3), special cases of which are the basic equations of gas dynamics (4), (5), (8);
– boundary conditions for solving the problem, presented in works [10–13];
– results of probe measurements of the electron energy distribution function in the external beam of the Hall effect thruster, presented in work [14].

The final results of the research are theoretical in nature and were obtained using identical mathematical transformations.

The basic parameters in gas dynamics are the moments of the particles velocity distribution function. The initial principle for writing the distribution function moments equations is the Boltzmann kinetic equation for velocity distribution function $f\left(\vec{r},\vec{v},t\right)$ [7]

$$\frac{\partial f\left(\vec{v}\right)}{\partial t}+\nabla\cdot\left(f\left(\vec{v}\right)\vec{v}\right)+\frac{q}{m}\nabla_v\cdot\left(f\left(\vec{v}\right)\left(\vec{E}+\vec{v}\times\vec{B}\right)\right)=\frac{\delta f\left(\vec{v}\right)}{\delta t},\qquad(1)$$

where t, $\vec{r}$, $\vec{v}$ – time, coordinate and velocity; q, m – particle charge and mass; ∇_v – the Hamiltonian operator in velocity space; $\vec{E}$, $\vec{B}$ – electric field strength and magnetic induction; $\delta f\left(\vec{v}\right)/\delta t$ – collision integral (change in the distribution function per unit time as a result of collisions).

Main gas dynamics parameters are the moments of the distribution function of *k*-th order $\mathbf{M}^{[k]}\left(\vec{r},t\right)$ – the *k*-th rank tensors

$$\mathbf{M}^{[k]}\left(\vec{r},t\right)=m\int\limits_0^\infty\int\limits_\Omega f\left(\vec{r},\vec{v},t\right)\vec{v}^{\lceil k\rceil}v^2dv\,d\Omega,\qquad(2)$$

where Ω – solid angle; $\vec{v}^{\lceil k\rceil}=\underbrace{\vec{v}\,\vec{v}...\vec{v}}_{k\,times}$ – iteration (tensor power, factor) of velocity [8].

Here:
– $\mathbf{M}^{[0]}=\rho_M$ – mass density;
– $\mathbf{M}^{[1]}=\vec{p}^{(V)}$ – mass flux density ≡ momentum density;
– $\mathbf{M}^{[2]}=\mathbf{\Pi}$ – momentum flux density, half of the trace of which $1/2\,Tr\,\mathbf{M}^{[2]}=\varepsilon^{(V)}$ – energy density;
– $\mathbf{M}^{[3]}=\mathbf{Q}$ – unnamed moment of the 3rd rank, half of the vector trace of which $1/2\,Tr\,\mathbf{M}^{[3]}=\vec{q}$ – energy flux density.

Multiplying each term (1) on $m\vec{v}^{\lceil k\rceil}$ and integrating over volume in velocity space it is possible to write the universal form of the distribution function moment equation [6] with the use of unified symbolism from [9]

$$\frac{\partial\mathbf{M}^{[k]}}{\partial t}+\nabla\cdot\mathbf{M}^{[k+1]}-k\frac{q}{m}\left\lfloor\mathbf{M}^{[k-1]}\vec{E}+\mathbf{M}^{[k]}\times\vec{B}\right\rfloor=\frac{\delta\mathbf{M}^{[k]}}{\delta t},\qquad(3)$$

where $\lfloor\ \ \rfloor$ – symbol of the symmetry operation [9].

So, the first few moments equations are:
– continuum equation ($k=0$)

$$\frac{\partial\rho_M}{\partial t}+\nabla\cdot\vec{p}^{(V)}=\frac{\delta\rho_M}{\delta t};\qquad(4)$$

– motion equation ($k=1$)

$$\frac{\partial\vec{p}^{(V)}}{\partial t}+\nabla\cdot\mathbf{\Pi}-\frac{q}{m}\left(\rho_M\vec{E}+\vec{p}^{(V)}\times\vec{B}\right)=\frac{\delta\vec{p}^{(V)}}{\delta t};\qquad(5)$$

– momentum flux equation ($k=2$)

$$\frac{\partial\mathbf{\Pi}}{\partial t}+\nabla\cdot\mathbf{Q}-2\frac{q}{m}\left\lfloor\vec{p}^{(V)}\vec{E}+\mathbf{\Pi}\times\vec{B}\right\rfloor=\frac{\delta\mathbf{\Pi}}{\delta t};\qquad(6)$$

– the 3rd moment equation

$$\frac{\partial\mathbf{Q}}{\partial t}+\nabla\cdot\mathbf{M}^{[4]}-3\frac{q}{m}\left\lfloor\mathbf{\Pi}\vec{E}+\mathbf{Q}\times\vec{B}\right\rfloor=\frac{\delta\mathbf{Q}}{\delta t},\qquad(7)$$

where $\mathbf{M}^{[4]}$ – unnamed the 4th order moment.

Here it can be seen the main problem of gas dynamics – the fundamental non-closure of moments equations set. Closing the system is only possible approximately using appropriate assumptions.

In the "classical" set, the equation of the 3rd moment is not written, and the energy equation is written instead of the momentum flux equation (6)

$$\frac{\partial\varepsilon^{(V)}}{\partial t}+\nabla\cdot\vec{q}-\frac{q}{m}\vec{p}^{(V)}\cdot\vec{E}=\frac{\delta\varepsilon^{(V)}}{\delta t}.\qquad(8)$$

The parameters of equations (4)–(8) can be represented as:

$$\vec{p}^{(V)}=\rho_M\vec{V};\qquad(9)$$

$$\mathbf{\Pi}=\rho_M\vec{V}^{\lceil 2\rceil}+\mathbf{P}=\rho_M\vec{V}^{\lceil 2\rceil}+\boldsymbol{\delta}\mathrm{P}+\boldsymbol{\pi};\qquad(10)$$

$$\varepsilon^{(V)}=\frac{\rho_M V^2}{2}+\frac{3}{2}P;\qquad(11)$$

$$\mathbf{Q}=\rho_M\vec{V}^{\lceil 3\rceil}+3\left\lfloor\vec{V}\mathbf{P}\right\rfloor+\mathbf{Q}_t;\qquad(12)$$

$$\vec{q}=\vec{V}\left(\frac{\rho_M V^2}{2}+\frac{3}{2}P\right)+\vec{V}\cdot\mathbf{P}=\vec{V}\left(\frac{\rho_M V^2}{2}+\frac{5}{2}P\right)+\vec{V}\cdot\boldsymbol{\pi}+\vec{q}_t;\qquad(13)$$

$$\mathbf{M}^{[4]}=\rho_M\vec{V}^{\lceil 4\rceil}+6\left\lfloor\vec{V}^{\lceil 2\rceil}\mathbf{P}\right\rfloor+4\left\lfloor\vec{V}\mathbf{Q}_t\right\rfloor+\boldsymbol{M}^{[4]},\qquad(14)$$

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where $\vec{V}$ – mass flux velocity; $\mathbf{P}$, P – pressure tensor and scalar; $\boldsymbol{\delta}$ – unitary tensor; $\boldsymbol{\pi}$ – viscosity tensor; $\mathbf{Q}_t$, $\boldsymbol{M}^{[4]}$ – unnamed static moments of the 3rd and 4th order; $\vec{q}_t$ – heat conduction.

Using the expressions above, it is possible to write the equation for the third static moment $\mathbf{Q}_t$

$$\frac{\partial \mathbf{Q}_t}{\partial t}+\nabla\cdot\left(\vec{V}\mathbf{Q}_t+\boldsymbol{M}^{[4]}\right)+ +3\left\lfloor \mathbf{Q}_t\cdot\nabla\vec{V}-\frac{q}{m}\mathbf{Q}_t\times\vec{B}-\frac{\mathbf{P}\nabla\cdot\mathbf{P}}{\rho_M}\right\rfloor=\frac{\delta\mathbf{Q}_t}{\delta t}. \tag{15}$$

The generalization of the Maxwell expression for $\boldsymbol{M}^{[4]}$ within the framework of the local thermodynamic equilibrium method is presented in the work [10]

$$\boldsymbol{M}^{[4]}=3\frac{\lfloor \mathbf{P}P\rfloor}{\rho_M}, \tag{16}$$

and, according to (15)

$$\frac{\partial\vec{q}_t}{\partial t}+\nabla\cdot\left(\vec{V}\vec{q}_t\right)+\vec{q}_t\cdot\nabla\vec{V}+\mathbf{Q}_t\overset{(2)}{\bullet}\nabla\vec{V}+\frac{3}{2}\nabla\cdot\left(\frac{P\mathbf{P}}{\rho_M}\right)+\nabla\cdot\left(\frac{\mathbf{P}\cdot\mathbf{P}}{\rho_M}\right)- -\left(\frac{3}{2}\frac{P\nabla\cdot\mathbf{P}}{\rho_M}+\frac{\mathbf{P}\cdot\left(\nabla\cdot\mathbf{P}\right)}{\rho_M}\right)-\frac{q}{m}\vec{q}_t\times\vec{B}=\frac{\delta\vec{q}_t}{\delta t}, \tag{17}$$

where $\overset{(n)}{\bullet}$ – symbol of multiple dot product [9].

The heat conduction equation known in gas dynamics is an approximate notation of the expression (17) neglecting viscosity and the changes in the heat conduction itself in space and time, presented in the first line (17) compared to the changes in collisions

$$\frac{5}{2}P\nabla\left(\frac{P}{\rho_M}\right)-\frac{q}{m}\vec{q}_t\times\vec{B}=\frac{\delta\vec{q}_t}{\delta t}. \tag{18}$$

Taking into account axial symmetry and approximately considering the magnetic field to be radial, it is possible to write:

$$\frac{5}{2}\frac{P_e}{m_e}k\frac{\partial T_e}{\partial r}=-\frac{q_{ter}}{\tau_e^{(q)}}, \tag{19}$$

$$\frac{5}{2}\frac{P_e}{m_e}k\frac{\partial T_e}{\partial x}-\frac{eB}{m_e}q_{te\phi}=-\frac{q_{tex}}{\tau_e^{(q)}}, \tag{20}$$

$$\frac{eB}{m_e}q_{tex}=-\frac{q_{te\phi}}{\tau_e^{(q)}}, \tag{21}$$

and

$$\frac{5}{2}\frac{P_e}{m_e}k\frac{\partial T_e}{\partial x}=-\frac{\left(1+\chi^2\right)q_{te\phi}}{\tau_e^{(q)}}, \tag{22}$$

where $\tau_e^{(q)}$ – effective relaxation time of heat conduction in collisions; χ – Hall parameter

$$\chi=\frac{eB}{m_e\tau_e^{(q)}}. \tag{23}$$

Taking into account the great free path time of electrons in the rarefied substance of EPCD and small mass of the electron, expressions (19) and (22) mean that the electrons temperature is practically constant in both directions, which is completely inconsistent with the results of measurements of the temperature distribution in the Hall effect thruster channel, presented, among other things, in works [3–5].

It should be noted, however, that at high potential barrier compared to the electron's temperature, the generalization (16) for $\boldsymbol{M}^{[4]}$ is inadmissible. The difference compared to dense gases is that mass and energy in this case are not transferred by all electrons, but only by those that are able to overcome the potential difference between a fixed point and the surface. The filling of this upper energy region occurs almost only under the action of the electric field and due to the non-mirror reflection from the potential barrier. The role of collisions in the volume is insignificant.

2.2. Kinetics, gas dynamics and compromise kinetic-fluid methods

Being not Maxwell one, electrons velocity distribution function in thrusters with close electron drift however is closed to isotropic one due to two factors: strong magnetic field and non-mirror reflection of electrons from Langmuir bound of plasma [10]. This means that the solution to the problem should be sought within the framework of a compromise kinetic-fluid model [6] using the angular moments $\mathbf{F}^{[k]}(v)$ of the distribution function

$$\mathbf{F}^{[k]}=\frac{1}{4\pi}\int f_e(\vec{v})i_v^{[k]}d\Omega, \tag{24}$$

such that the total moments $\mathbf{M}_e^{[k]}$ are equal to

$$\mathbf{M}_e^{[k]}=4\pi m_e\int_0^\infty \mathbf{F}^{[k]}(v)v^{k+2}dv. \tag{25}$$

The universal notation for the angular moment equation is [6]

$$\frac{\partial\mathbf{F}^{[k]}}{\partial t}+v\nabla\cdot\mathbf{F}^{[k+1]}+k\frac{e}{m_e}\left\lfloor\frac{1}{v}\mathbf{F}^{[k-1]}\vec{E}+\mathbf{F}^{[k]}\times\vec{B}\right\rfloor- -\frac{e}{m_e}\vec{E}\cdot\frac{1}{v^{k+2}}\frac{\partial}{\partial v}\left(v^{k+2}\mathbf{F}^{[k+1]}\right)=\frac{\delta\mathbf{F}^{[k]}}{\delta t}. \tag{26}$$

For example, the Maxwell distribution for a substantially subsonic flow of electrons

$$f(\vec{v})\approx C\exp\left(-\frac{m_e v^2}{2kT_e}\right)\left(1+\frac{m_e\vec{v}\cdot\vec{V}}{kT_e}\right), \tag{27}$$

means

$$\frac{\mathbf{F}^{[1]}(v)}{\mathbf{F}^{[0]}(v)}=\frac{\vec{f}_1(v)}{f_0(v)}=\frac{m_e\vec{V}}{3kT_e}v, \tag{28}$$

i. e. the input of all electrons to the transfer of mass and energy is proportional to the modulus of their velocity, which contradicts the above-mentioned features.

Taking these features into account, the problem can be solved in the following sequence.

The equations for the 0th and 1st angular moments in stationary form are [6]:

$$v\nabla\cdot\vec{f}_1-\frac{e}{m_e}\vec{E}\cdot\frac{1}{v^2}\frac{\partial}{\partial v}\left(v^2\vec{f}_1\right)=\frac{\delta f_0}{\delta t}, \tag{29}$$

$$v\nabla\cdot\mathbf{F}^{[2]}+\frac{e}{m_e}\left(\frac{1}{v}f_0\vec{E}+\vec{f}_1\times\vec{B}\right)-\frac{e}{m_e}\vec{E}\cdot\frac{1}{v^3}\frac{\partial}{\partial v}\left(v^3\mathbf{F}^{[2]}\right)=\frac{\delta\vec{f}_1}{\delta t}. \tag{30}$$

3. Results and Discussion

3.1. Formation of equations set

The main component of the electrons mass flux density in the EPCD is its azimuth projection, which allows to take into account only the radial-azimuth one in the non-diagonal components of $\mathbf{F}^{[2]}$. So, neglecting collisions in the volume and in the approximation

$$\mathbf{F}^{[2]} = \boldsymbol{\delta}/3\, f_0 + \left(i_r i_\phi + i_\phi i_r\right) f^{(r\phi)}, \tag{31}$$

it is possible to write

$$\frac{v}{3}\left(\nabla f_0 - \frac{e}{m_e}\vec{E}\frac{1}{v}\frac{\partial f_0}{\partial v}\right) + i_\phi \frac{v}{r^2}\frac{\partial}{\partial r}\left(f^{(r\phi)} r^2\right) + \frac{e}{m_e}\vec{f}_1 \times \vec{B} = 0. \tag{32}$$

Equation (32) is obtained by taking into account the following property

$$\frac{1}{3}\frac{1}{v^3}\frac{\partial}{\partial v}\left(v^3 f_0\right) - \frac{1}{v} f_0 = \frac{1}{3}\frac{1}{v}\frac{\partial f_0}{\partial v}. \tag{33}$$

Selecting the axial and radial projections of the 1st moment

$$\vec{f}_1^{(0)} = \vec{f}_1 - i_\phi f_\phi = i_x f_x + i_r f_r, \tag{34}$$

it is possible to write:

$$v\nabla \cdot \vec{f}_1^{(0)} - \frac{e}{m_e}\vec{E} \cdot \frac{1}{v^2}\frac{\partial}{\partial v}\left(v^2 \vec{f}_1^{(0)}\right) = \frac{\delta f_0}{\delta t}, \tag{35}$$

$$\frac{v}{3}\left(\nabla f_0 - \frac{e}{m_e}\vec{E}\frac{1}{v}\frac{\partial f_0}{\partial v}\right) - i_x \frac{e}{m_e} f^{(\phi)} B = \frac{\delta \vec{f}_1^{(0)}}{\delta t}, \tag{36}$$

$$\frac{v}{r^2}\frac{\partial}{\partial r}\left(f^{(r\phi)} r^2\right) + \frac{e}{m_e} f^{(x)} B = \frac{\delta f^{(\phi)}}{\delta t}. \tag{37}$$

Using the results quasi-one-dimensional mathematical model [10] and partial kinetic solution for electrons motion [11] one can write:

$$\frac{2}{3} f_s^{(r\phi)} = \frac{\xi_n}{4}\eta_e^{(p)} f^{(\phi)}, \tag{38}$$

$$f^{(\phi)} = -\frac{4R}{\xi_n \eta_e^{(p)}}\frac{eB}{m_e v} f^{(x)}, \tag{39}$$

$$v\frac{\partial f^{(x)}}{\partial x} - \frac{e}{m_e}E_x \frac{1}{v^2}\frac{\partial}{\partial v}\left(v^2 f^{(x)}\right) + \frac{\xi_n}{R} v f_0 = \frac{v f_0}{\lambda_i}, \tag{40}$$

$$\frac{\partial f_0}{\partial x} - \frac{e}{m_e}E_x \frac{1}{v}\frac{\partial f_0}{\partial v} + \frac{12R}{\xi_n \eta_e^{(p)}}\left(\frac{eB}{m_e v}\right)^2 f^{(x)} = 0, \tag{41}$$

where λ_i – free path length according to ionization.

Expressions (40), (41) can be rewritten as:

$$\frac{\partial}{\partial x}\left(v^2 f^{(x)}\right) - \frac{e}{m_e}E_x \frac{1}{v}\frac{\partial}{\partial v}\left(v^2 f^{(x)}\right) = \left(\frac{1}{\lambda_i} - \frac{\xi_n}{R}\right) v^2 f_0, \tag{42}$$

$$\frac{\partial f_0}{\partial x} - \frac{e}{m_e}E_x \frac{1}{v}\frac{\partial f_0}{\partial v} = -\frac{12R}{\xi_n \eta_e^{(p)}}\left(\frac{eB}{m_e v^2}\right)^2 v^2 f^{(x)}, \tag{43}$$

and must be supplemented with appropriate boundary conditions.

3.2. Specific of electrons flow limited by high potential barrier

The solution about electrons energy distribution function in the volume limited by potential barrier inside boundary bipolar layer is obtained in the paper [12]. The result of solution is presented by Fig. 1. The comparison with Maxwellian distribution shows the great difference of electrons amount in the area upper the barrier.

Instead of variables x and v, it is convenient to solve the problem in variables:

$$\varepsilon = \frac{m_e\left(v^2 - v_p^2\right)}{2}, \tag{44}$$

$$v_p^2 = \frac{2e}{m_e}\left(\phi - \phi_L - \Delta\phi\right), \tag{45}$$

where ϕ – the potential at a point in the volume; ϕ_L – the potential on the boundary with bipolar layer; $\Delta\phi$ – the potential drop inside the bipolar layer (Fig. 1).

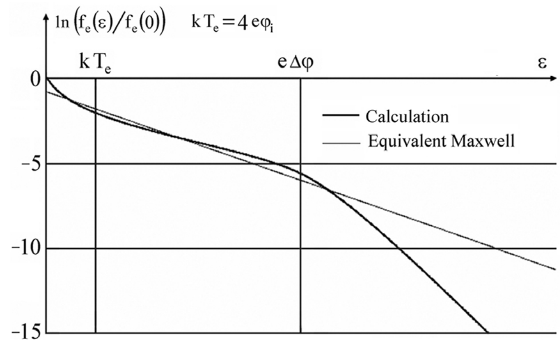

Fig. 1. Electron energy distribution function [12]

Unfortunately, the typical dimensions of Hall effect thrusters do not allow for the measurement of electron energy distribution directly in the thruster channel. The results of measurements in the external beam are presented in [14] (Fig. 2).

The Maxwell distribution graph and linear approximation are added to the figure. The difference from Fig. 1 is that a multiplier $\sqrt{\varepsilon}$ was not introduced into the distribution function – therefore, the picture in the low-energy region is more informative than in Fig. 2. But this difference becomes less significant at $\varepsilon \geq 3kT_e$. The calculation by linear approximation gives the same results as calculation by our method, as evidenced by the break of the graph near $e\Delta\phi = 5kT_e$, as in Fig. 1. It is characteristic that in [14] the results of measurements in the outer beam of the thruster are given, where there are no walls and, accordingly, secondary electron-electron emission, which has been tried to "correct" the results of calculations using the Maxwell distribution absent in the EPCD for many years [3–5].

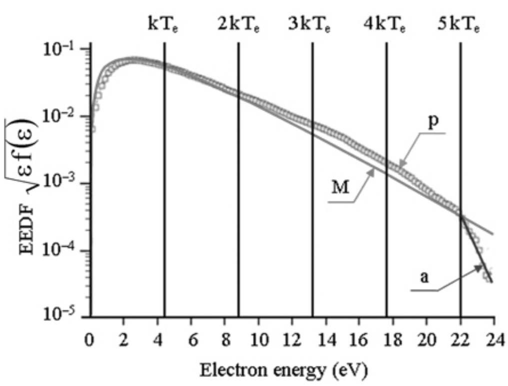

Fig. 2. Results of measurements of the electron energy distribution function: p – experiment; M – Maxwell; a – approximation

Generalizing the boundary conditions obtained in work [13], it is possible to introduce new variables:

$$f_0(x, v) = \Phi_0(x, \varepsilon), \tag{46}$$

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$$v^2 f^{(x)}(x,v)=\frac{\varepsilon}{2m_e}\Phi_x(x,\varepsilon). \tag{47}$$

Taking into account the property

$$\frac{\partial\varepsilon}{\partial x}-\frac{e}{m_e}E_x\frac{1}{v}\frac{\partial\varepsilon}{\partial v}=0, \tag{48}$$

it is possible to write:

$$\frac{\partial\Phi_0}{\partial x}=-\frac{3R}{\xi_n\eta_e^{(p)}}\left(\frac{eB}{m_e v}\right)^2\frac{2\varepsilon}{m_e v^2}\Phi_x, \tag{49}$$

$$\frac{\partial\Phi_x}{\partial x}=4\left(\frac{1}{\lambda_i}-\frac{\xi_n}{R}\right)\frac{m_e v^2}{2\varepsilon}\Phi_0. \tag{50}$$

In the new variables x and ε, the velocity v is a function

$$v(x,\varepsilon)=\sqrt{\frac{2}{m_e}\left(e\left(\phi(x)-\phi_L-\Delta\phi\right)+\varepsilon\right)}. \tag{51}$$

It is shown in work [13] that under HET and PIT-R conditions when operating on xenon the relations take place:

$$e\Delta\phi\approx 5kT_e, \tag{52}$$

$$\langle\varepsilon\rangle\approx kT_e, \tag{53}$$

where $\langle\varepsilon\rangle$ – the average value of ε for electrons overcoming the barrier.

Also the mean difference $\phi-\phi_L$ is about

$$e(\phi-\phi_L)_{\max}\approx 0.5kT_e. \tag{54}$$

In accordance with (51)–(54) the quantities ε and $e(\phi-\phi_L)$ represent a small contributions to the velocity v, which allows to consider

$$v\approx const. \tag{55}$$

So, it is possible to combine (49) and (50)

$$\frac{\partial^2\Phi_x}{\partial x^2}\approx-\frac{12R}{\eta_e^{(p)}}\left(\frac{1}{\xi_n\lambda_i}-\frac{1}{R}\right)\left(\frac{eB}{m_e v}\right)^2\Phi_x. \tag{56}$$

Thus, the coefficient on the right side of the expression (56) is a function of only the coordinate, as a result of which the solution of the equation (56) can be written as

$$\Phi_x(x,\varepsilon)=\Phi_a(\varepsilon)S\left(\sqrt{\frac{12R}{\eta_e^{(p)}}\left(\frac{1}{\xi_n\lambda_i}-\frac{1}{R}\right)}\frac{eB}{m_e v_p}x\right), \tag{57}$$

where

$$\frac{\partial}{\partial\varepsilon}S\left(\sqrt{\frac{12R}{\eta_e^{(p)}}\left(\frac{1}{\xi_n\lambda_i}-\frac{1}{R}\right)}\frac{eB}{m_e v_p}x\right)\approx 0. \tag{58}$$

Using (44), (47) in this case it is possible to write

$$v^2 f^{(x)}(x,v)=\frac{1}{4}f_a^{(x)}S\left(\sqrt{\frac{12R}{\eta_e^{(p)}}\left(\frac{1}{\xi_n\lambda_i}-\frac{1}{R}\right)}\frac{eB}{m_e v_p}x\right)\times$$
$$\times\left(v^2-v_p^2\right)\exp\left(-\frac{m_e\left(v^2-v_p^2\right)}{2kT_e^+}\right), \tag{59}$$

where $f_a^{(x)}$ – scale factor.

Taking into account (25) it follows from (59) for the axial projections of the mass and energy flux densities:

$$p_{ex}^{(V)}(x)=4\pi m_e\int_{v_p}^{\infty}f^{(x)}(x,v)v^3dv, \tag{60}$$

$$q_{ex}(x)=2\pi m_e\int_{v_p}^{\infty}f^{(x)}(x,v)v^5dv, \tag{61}$$

and, finally

$$\frac{q_{ex}}{p_{ex}^{(V)}}=\frac{\int_0^{\infty}\exp\left(-\frac{m_e v^2}{2kT_e^+}\right)\left(v^2+v_p^2\right)v^3dv}{2\int_0^{\infty}\exp\left(-\frac{m_e v^2}{2kT_e^+}\right)v^3dv}=e(\phi-\phi_p)+2kT_e^+, \tag{62}$$

where $T_e^+\approx 0.5T_e$ – effective temperature of electrons in the upper part of the energy spectrum (Fig. 1).

As a result it is possible to write for any point in the volume

$$\frac{q_{ex}}{p_{ex}^{(V)}kT_e}\approx const, \tag{63}$$

and, taking into account that in radial direction $S=const$

$$\frac{q_{er}}{p_{er}^{(V)}kT_e}\approx const. \tag{64}$$

The values of the constants in the right-hand parts of expressions (63) and (64) are quantitatively equal to those in the electron flow from the plasma to the boundary layer, determined in [12, 13].

In the final form, equations (63), (64) take the form

$$\frac{q_{en}}{m_e V_{en}kT_e}=e\Delta\phi_n+\mu_{sn}, \tag{65}$$

where m_e, T_e – electron mass and temperature; V_{en}, q_{en} – projections of the mass flux velocity and energy flux density of electrons in the direction of the boundary n; $e\Delta\phi_n$, μ_{sn} – the height of the potential barrier and the residual energy of an electron reaching the boundary n, found using the compromise kinetic-fluid model.

The only approximation used in the research is expression (54), which introduces an error of ±5% into the overall result (65).

Substitution of (65) into the electron's energy equation (8) allows to lose the equations set of the mathematical model of processes in propulsion devices with closed electron drift in the presence of a potential barrier in the boundary layer with the plasma.

3.3. Discussion

The results presented in this paper:

– explain the specific of electrons energy transport in EPCD volume;

– differ compared with already known effects lays in the presence of potential barrier in side boundary bipolar layer, when previously known expressions assuming the participation of the entire electron spectrum in this process lead to the conclusion about constant electron temperature in the device volume, which is completely inconsistent with the measurement results;

– can be applied and are reproducible in the case of subsonic electron flow, which is well known feature of the plasma in all the types of electric propulsion devices but not in electronic devices with visible difference between ions and electrons population.

The practical significance of this work lies in the possibility of formulating consistent closed systems in mathematical models of various types of EPCD. Further analysis of the role of electron-atom and electron-ion collisions in the "extraction" of electrons from the main part of the spectrum, trapped by potential barriers, into the energy range corresponding to their mass and energy fluxes beyond the plasma boundary appears necessary.

A promising future research opportunity is to analyze the possible factors of electrons transport from energy area below the potential barrier upper of it.

4. Conclusions

1. The obtained results:
– were obtained being connected with boundary conditions for the electrons' energy flux density;
– demonstrate the coincidence of the proportion between the energy and mass flows of electrons in the plasma volume and in the boundary bipolar layer;
– allow to close the system of equations of the mathematical model of processes in EPCD;
– are in quantitative correspondence with measurements in terms of the height of the potential barrier and the population of the spectral region above it.

2. The difference from the characteristics of dense gases with a moderate change in potential energy in the entire volume between the boundaries is due to the fact that in the rarefied plasma of EPCD in the almost absence of collisions and in the presence of high potential barriers for electrons, the transfer of mass and energy is carried out only by a small fraction of electrons in the upper part of the energy spectrum when the majority of electrons (with energy below the barrier) remain motionless, and the balance of the number of particles is determined by ionization on the one hand and their "ejection" into the upper part of the spectrum by the electric field and, in part, by collisions on the other hand.

Conflict of interest

The author declares that he has no conflict of interest in relation to this research, whether financial, personal, authorship or otherwise, that could affect the research and its results presented in this paper.

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Data availability

The manuscript has no associated data.

Use of artificial intelligence

The author confirms that he did not use artificial intelligence technologies when creating the current work.

Authors' contributions

The sole author, ***Shahram Roshanpour***, is responsible for the entire content of the manuscript, including the conceptualization, methodology design, data collection, experimental execution, formal analysis, results interpretation, writing the original draft, and final manuscript review and editing.

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