AN ANALYTICAL APPROCH FOR SOLVING FRACTIONAL FUZZY OPTIMAL CONTROL PROBLEM WITH FUZZY INITIAL CONDITIONS

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Abstract. A fractional – fuzzy optimal control problem is an optimal control problem in which it is governed by a fuzzy system of fractional differential equation. The aim of this paper is to introduce an analytically solution for such Bolza problems when the initial state is also fuzzy. For this purpose, first the problem is turned to two fractional optimal control problems by concept of α-cut and complex numbers. Then, we apply a new method to solve these fractional optimal control problems, analytically by applying a new Riccati differential equation determined from PMP. Indeed this Riccati equation transfer each mentioned fractional optimal control problem to a fractional differential system. We show that if the new system has close solution, one is able to obtain the analytical solution of the fractional – fuzzy optimal control problems. A numerical simulation based on the new method is presented for different values of α and fractional order and the results are compered. In the last section, a numerical example of fractional-fuzzy optimal control problem is solved by the new method for different α and β; and compared with the exact state; also, they are shown in figures for each cases.

Key words: Fractional differential equation, optimal control, fuzzy.

Introduction. Many real events and dynamical systems have uncertainty in their inputs, outputs and manners; we know that fuzziness is a very adequate tool to present the suitable kind of uncertainly phenomena in the real world. In this regard, by involving fuzziness in the optimal control theory, problems can be demonstrated better with control parameters in real world as physical models and dynamical systems. In the last years, fractional calculus plays very important roles in mathematics, mechanics, and other subject. Many dynamical systems and events have much better efficiency when they planned by using fractional differential equations.

We know that in optimal control theory, if a fuzzy differential equation in fractional order contains a control variable, then we have a fractional fuzzy optimal control problem (FFOCP). Zhu [20] applied Bellman's optimal principle to make optimality conditions for fuzzy optimal control problems; Diamond and Kloeden [16] discussed on existence the solution of such control systems. Then, Park et.al [9] obtained the sufficient conditions for fuzzy control systems. Filev and Angelove [4] had solved fuzzy optimal control problems of nonlinear system with fuzzy mathematical programing. Z. Qin [22] solved the time-homogeneous fuzzy optimal control problems, discounted objective function. In [3] by considering the generalized differentiability, authors used new solutions for fuzzy two point boundary value problems for Hukuhara differentiability. Georgiou et.al [5] discussed *n*th-order fuzzy differential equations with initial value conditions. Nieto et.al [10] found numerical methods for solving fuzzy differential equations. Agrawal [1], used the Lagrange multipliers technique, to obtain necessary conditions for optimality of fuzzy optimal control problems. To continue, in this paper, we consider a fractional fuzzy control problem and turn it to two fractional control problems and then using fractional Pontryagin Maximum Principle to solve it.

Consider the fractional-fuzzy control problem as following:

$$\text{Min: } \int_a^b f_0(t,\tilde{x}(t),\tilde{u}(t))dt$$

$$\text{S. to: } \begin{cases} ({}_{a+}D^{\beta}\tilde{x})(t) = f(t,\tilde{x}(t),\tilde{u}(t)); \\ \tilde{x}(a) = \tilde{x}_0 \quad , \end{cases} \tag{1}$$

where $t \in (a,b) \subseteq \mathbb{R}$, $\tilde{x}$ is a fuzzy bounded trajectory, $\tilde{x}_0$ is a fuzzy initial condition, $\tilde{u}$ is fuzzy control variable, f and f_0 are two given continues functions respect to $t, \tilde{x}$ and $\tilde{u}$, here $({}_{a+}D^{\beta}\tilde{x})(t)$ denotes the left Riemann-Liouville derivative at order $\beta \in (0,1)$. We remind that Problem (1), that is fuzziness respect to x and u and also governed by a differential equation with fractional order, is a fractional – fuzzy optimal control problem. The aim of this paper, is to find a fuzzy solution for these kind of problems. This paper is organize as in section 2.

Preliminaries and notations

In this section, first, we remind some necessary definitions and theorems, which are required for fuzzy and fractional calculations.

Definition 2.1: Signify E^1 as the set of all functions $x(t)$ that satisfy in the following conditions:

(i) x is normal, i.e. there exist $t \in \mathbb{R}$, such that $x(t) = 1$;

(ii) x is fuzzy convex, i.e. $\forall\, s,t \in \mathbb{R}$ and $\lambda \in [0,1]$, $x(\lambda s + (1-\lambda)t) \geq \min\{x(s), x(t)\}$;

(iii) x is upper semi-continuous;

(iv) $cl\{s \in \mathbb{R} | x(s) > 0\}$, is compact in $\mathbb{R}$.

Definition 2.2: The α-level set of a fuzzy number $x \in E^1$ where $0 \leq \alpha \leq 1$, is denoted by x_α and is defined as:

$$x_\alpha = \begin{cases} \{s \in \mathbb{R} \mid x(s) \geq \alpha\}, 0 < \alpha \leq 1; \\ cl\,\{s \in \mathbb{R} \mid x(s) > 0\}, \quad \alpha = 0. \end{cases} \quad (2)$$

If $x \in E^1$, then x is fuzzy convex, so x_α is closed and bounded in $\mathbb{R}$, i.e. $x_\alpha \equiv [\underline{x}_\alpha, \overline{x}_\alpha]$, where $\underline{x}_\alpha = \inf\{s \in \mathbb{R} \mid x(s) \geq \alpha\} > -\infty$ and $\overline{x_\alpha} = \sup\{s \in \mathbb{R} \mid x(s) \geq \alpha\} < \infty$.

Lemma 2.3: Let $I = [0,1]$ and assume that $a: I \to \mathbb{R}$ and $b: I \to \mathbb{R}$ satisfy the following conditions:

(i) a and b are bounded non-decreasing functions;

(ii) $a(1) \leq b(1)$;

(iii) For $0 < k \leq 1$, $\lim_{\alpha \to k^-} a(\alpha) = a(k)$ and $\lim_{\alpha \to k^-} b(\alpha) = b(k)$.

(iv) $\lim_{\alpha \to 0^+} a(\alpha) = a(0)$ and $\lim_{\alpha \to 0^+} b(\alpha) = b(0)$.

Then, $\eta: I \to \mathbb{R}$ defined by $\eta(x) = \sup\{\alpha \mid a(\alpha) \leq x \leq b(\alpha)\}$ is a fuzzy number with parameterization given by $\{(a(\alpha), b(\alpha), \alpha) \mid 0 \leq \alpha \leq 1\}$; moreover, if $\hat{\eta}: I \to \mathbb{R}$ is any fuzzy number with parameterization given by $\{(\hat{a}(\alpha), \hat{b}(\alpha), \alpha) \mid 0 \leq \alpha \leq 1\}$, then, functions $\hat{a}(\alpha)$ and $\hat{b}(\alpha)$ satisfy the above conditions (i) - (iv).

Proof: see [16].

Lemma 2.4: Assume each entry of the vector x be a fuzzy number at the time instant t where [8, 14]:

$$x_\alpha^k = [\underline{x}_\alpha^k, \overline{x}_\alpha^k], \qquad k = 1,2, \dots, n. \quad (3)$$

Then, the evaluation of the system:

$$\begin{cases} \dot{\tilde{x}}(t) = A \odot \tilde{x}(t); \\ \tilde{x}(t_0) = \tilde{x}_0, \end{cases} \quad (4)$$

where $\tilde{x}$ is a fuzzy function, $\tilde{x}_0$ is a fuzzy initial condition, $A = [a_{ij}]_{m \times n}$, $a_{ij} \in \mathbb{R}$ and $\dot{\tilde{x}} = \frac{dx}{dt} = [\frac{dx_1}{dt}, \frac{dx_2}{dt}, \dots, \frac{dx_n}{dt}]^T$, can be described by $2n$ differential equations for the endpoints of the intervals (3). The equations of the intervals are as follows:

$$\begin{cases} \dot{\underline{x}}_\alpha^k(t) = \min\{(Ay)^k : y^j \in [\underline{x}_\alpha^j(t), \overline{x}_\alpha^j(t)]\}; \\ \dot{\overline{x}}_\alpha^k(t) = \max\{(Ay)^k : y^j \in [\underline{x}_\alpha^j(t), \overline{x}_\alpha^j(t)]\}; \\ \underline{x}_\alpha(t_0) = \underline{x}_{\alpha 0}; \\ \overline{x}_\alpha(t_0) = \overline{x}_{\alpha 0}, \end{cases} \quad (5)$$

where $(Ay)^k := \sum_{j=1}^n a_{kj}\, y^j$ is the kth row of Au.

Proof: [14].

Since the vector field in (1) is linear, the following rule applies in (5):

$$\dot{\underline{x}}_\alpha^k(t) = \sum_{j=1}^n a_{kj} w^j,$$

where

$$w^j = \begin{cases} \underline{x}_\alpha^j(t), & a_{kj} \geq 0; \\ \overline{x}_\alpha^j(t), & a_{kj} < 0 \end{cases}$$

and

$$\dot{\overline{x}}_\alpha^k(t) = \sum_{j=1}^n a_{kj} z^j,$$

where

$$z^j = \begin{cases} \underline{x}_\alpha^j(t), & a_{kj} < 0; \\ \overline{x}_\alpha^j(t), & a_{kj} \geq 0. \end{cases}$$

Here, we have a characterization for an important class of fuzzy controlled system. Consider the following fuzzy linear controlled system with fuzzy boundary condition:

$$\begin{cases} \dot{\tilde{x}}(t) = A \odot \tilde{x}(t) \oplus C \odot \tilde{u}(t); \\ \tilde{x}(t_0) = \tilde{x}_0, \end{cases}$$

As indicated in [13], it is possible to represent a fuzzy number in a more compact form by moving to the field of complex number by defining new complex variables as follows;

$$x_\alpha^k = \underline{x}_\alpha^k(t) + i\overline{x}_\alpha^k(t), \qquad k = 1,2, \dots, n. \quad (6)$$

Now, we have the following theorem in which its proof can be find in [14].

Theorem 2.5: Let A and C be $n \times n$ and $n \times m$ matrices respectively. Then for a given $\tilde{x}_0$, the fuzzy controlled system

$$\begin{cases} \dot{\tilde{x}}(t) = A \odot \tilde{x}(t) \oplus C \odot \tilde{u}(t); \\ \tilde{x}(t_0) = \tilde{x}_0, \end{cases} \quad (7)$$

has the following solution:

$$\begin{cases} \dot{\underline{x}}_\alpha(t) + i\dot{\overline{x}}_\alpha(t) = B(\underline{x}_\alpha(t) + i\overline{x}_\alpha(t)) + D(\underline{u}_\alpha(t) + i\overline{u}_\alpha(t)); \\ \underline{x}_\alpha(t_0) + i\overline{x}_\alpha(t_0) = \underline{x}_{\alpha 0} + i\overline{x}_{\alpha 0}, \end{cases} \tag{8}$$

where the elements of matrices B and D are determined from those of A and C as:

$$b_{ij} = \begin{cases} ea_{ij}, & a_{ij} \geq 0; \\ ga_{ij}, & a_{ij} < 0; \end{cases} \qquad d_{ij} = \begin{cases} ec_{ij}, & c_{ij} \geq 0; \\ gc_{ij}, & c_{ij} < 0, \end{cases} \tag{9}$$

that for every $a + bi \in \mathbb{C}$ (the complex numbers field), the function e and g are defined as:

$$e: a + bi \rightarrow a + bi, \qquad g: a + bi \rightarrow b + ai. \tag{10}$$

Definition 2.6: For a given $\alpha \in [0,1]$ and arbitrary $\tilde{x} = (\underline{x}_\alpha, \overline{x}_\alpha)$, $\tilde{y} = (\underline{y}_\alpha, \overline{y}_\alpha)$ if k be a real number, we define addition $\tilde{x} + \tilde{y}$, subtraction $\tilde{x} - \tilde{y}$ and scalar multiplication by k as [8, 13]:

$$\tilde{x} + \tilde{y} = (\underline{x}_\alpha + \underline{y}_\alpha, \overline{x}_\alpha + \overline{y}_\alpha); \qquad \tilde{x} - \tilde{y} = (\underline{x}_\alpha - \underline{y}_\alpha, \overline{x}_\alpha - \overline{y}_\alpha);$$

$$k \odot \tilde{x} = \begin{cases} (k\underline{x}_\alpha, k\overline{x}_\alpha), & k \geq 0; \\ (k\overline{x}_\alpha, k\underline{x}_\alpha), & k < 0. \end{cases}$$

Note that as mentioned in many references like [7, 13] we can rewrite any fuzzy number by an interval using α-level parameterization. Assume that $\tilde{x} = (p, q, r)$ be a triangular fuzzy number; one can show this number by α-level parameterization, as follow [14]:

$$\tilde{x} = [q\alpha + p(1-\alpha), q\alpha + r(1-\alpha)], \qquad \alpha \in [0,1]. \tag{11}$$

Fuzzy Riemann-Liouville differential

Regarding the governing system of our optimal control problem (1), this section, is devoted to present the definition of fuzzy Riemann-Liouville integrals and derivatives by Hukuhara difference. Memorize that $C^F[a, b]$ is the space of all continuous fuzzy-valued functions on $[a, b]$ and $L^F[a, b]$ is the space of all Lebesque integrable fuzzy-valued functions on the bounded interval $[a, b] \subset \mathbb{R}$.

Definition 3.1: Let $f \in C^F[a, b] \cap L^F[a, b]$. Then fuzzy Riemann-Liouville integral of fuzzy-valued function f is defined as following [2, 19]:

$$(I^\beta_{a+} f)(x) = \frac{1}{\Gamma(\beta)} \int_a^x \frac{f(t)dt}{(x-t)^{1-\beta}}, \quad 0 < \beta \leq 1; \tag{12}$$

$$(I^\beta_{b-} f)(x) = \frac{1}{\Gamma(\beta)} \int_x^b \frac{f(t)dt}{(x-t)^{1-\beta}}, \quad 0 < \beta \leq 1; \tag{13}$$

where $(I^\beta_{a+} f)(x)$ and $(I^\beta_{b-} f)(x)$ are called respectively the left-sided and the right-sided Riemann-Liouville integral of the function f of order β.

Remark that the α-cut representation of fuzzy-valued function $f \in C^F[a, b] \cap L^F[a, b]$ is shown by $f(x; \alpha) = [\underline{f}(x; \alpha), \overline{f}(x; \alpha)]$ for $0 \leq \alpha \leq 1$, where $\underline{f}(x; \alpha)$ and $\overline{f}(x; \alpha)$ are defined as lower and upper bounds of α-level set of f, respectivly.

Theorem 3.1: Let $f \in C^F[a, b] \cap L^F[a, b]$ is a fuzzy-valued function. The fuzzy Riemann-Liouville integral of a fuzzy-valued function f can be expressed as follow:

$$(I^\beta_{a+} f)(x; \alpha) = [(I^\beta_{a+} \underline{f})(x; \alpha), (I^\beta_{a+} \overline{f})(x; \alpha)], \; 0 \leq \alpha \leq 1, \; 0 < \beta \leq 1; \tag{14}$$

where

$$(I^\beta_{a+} \underline{f})(x; \alpha) = \frac{1}{\Gamma(\beta)} \int_a^x \frac{\underline{f}(t; \alpha)dt}{(x-t)^{1-\beta}}; \qquad (I^\beta_{a+} \overline{f})(x; \alpha) = \frac{1}{\Gamma(\beta)} \int_a^x \frac{\overline{f}(t; \alpha)dt}{(x-t)^{1-\beta}}$$

Proof: See [2].

We remind that $(I^\beta_b f)(x; \alpha)$ can be defined like (14) as well.

Now, we are going to define the fuzzy Riemann-Liouvelle derivation of order $0 < \beta \leq 1$ for fuzzy-valued function f.

Definition 3.2: Let $\in C^F[a, b] \cap L^F[a, b]$, $x_0 \in (a, b) \subseteq \mathbb{R}$ and denote: $\Phi(x) \equiv \frac{1}{\Gamma(1-\beta)} \int_a^x \frac{f(t)}{(x-t)^\beta}$. Then f is called Riemann-Liouville H-differentiable of order $0 < \beta \leq 1$ at x_0, if there exist an element $(D^\beta_{a+} f)(x_0) \in E$, such that for sufficiently small $h > 0$, we have [19]:

$$(D^\beta_{a+} f)(x) = \lim_{h \to 0^+} \frac{\Phi(x_0 + h) \ominus \Phi(x_0)}{h} = \lim_{h \to 0^+} \frac{\Phi(x_0) \ominus \Phi(x_0 - h)}{h}$$

Theorem 3.2: Let $f \in C^F[a, b] \cap L^F[a, b]$, $x_0 \in (a, b)$ and $0 < \beta \leq 1$, then:

$$(D^\beta_{a+} f)(x; \alpha) = [(D^\beta_{a+} \underline{f})(x; \alpha), (D^\beta_{a+} \overline{f})(x; \alpha)], \qquad 0 \leq \alpha \leq 1;$$

also, we have the similar results for $(D^\beta_b f)(x; \alpha)$.

Proof: See [2].

Due to our main goal about fractional – fuzzy optimal control problems, in the sequence we discuss about the fractional Pontryagin's systems. Consider the following fractional optimal control problem:

$$\text{Min: } S(x(t_1), t_1) + \int_{t_0}^{t_1} f_0(t, x(t), u(t))dt \qquad (15\_A)$$

$$\text{S. to: } \begin{cases} (D_{a+}^{\beta} x)(t) = f(t, x(t), u(t)); \\ x(t_0) = A, \end{cases}$$

where A is a given real number. A necessary condition for (x^*, u^*) to be a solution of (15_A) is that there exist a function λ such that the following fractional Pontryagin's system holds [12, 17, 18]:

$$\begin{cases} D_{a+}^{\beta} x = \frac{\partial H}{\partial \lambda}(x, u, \lambda, t); \\ D_{b-}^{\beta} w = \frac{\partial H}{\partial x}(x, u, \lambda, t); \\ \frac{\partial H}{\partial u}(x, u, \lambda, t) = 0; \\ (x(t_0), \lambda(t_1)) = (A, 0), \end{cases} \qquad (15\_B)$$

where $H(x, u, \lambda, t) = f_0(t, x, u) + \lambda f(x, u, t)$ is the Hamiltonian function of (15). The states problem for fixed final point and free final point are $\delta x(t_1) = 0$ and $(\frac{\partial s}{\partial x} - \lambda)(t_1) = 0$, respectivly.

Fractional Fuzzy Optimal Control Problem

Based on the above discussions, we are going to propose a solution method for FFOCPs. Consider the following fractional-fuzzy optimal control problem:

$$\text{Min} \int_a^b f_0(t, \tilde{x}(t), \tilde{u}(t))dt \qquad (16)$$

$$\text{S. to:} \begin{cases} (D_{a+}^{\beta} \tilde{x})(t) = f(t, \tilde{x}(t), \tilde{u}(t)); \\ \tilde{x}(a) = \tilde{x}_0 = (p, q, r), \end{cases}$$

where the initial condition, $\tilde{x}_0 = (p, q, r)$ is a triangular fuzzy number and $0 < \beta \le 1$.

By using the concept of α-cut, Theorem 3.2 and parameterization of a fuzzy number, for each $0 \le \alpha \le 1$ we can write problem (16) in complex space as follows:

$$\text{Min: } \int_a^b f_0(t, \underline{x}(t; \alpha), \underline{u}(t; \alpha)) + i f_0(t, \overline{x}(t; \alpha), \overline{u}(t; \alpha))dt$$

S.to: (17)

$$\begin{cases} (D_{a+}^{\beta} \underline{x})(t; \alpha) + i(D_{a+}^{\beta} \overline{x})(t; \alpha) = f(t, \underline{x}(t; \alpha), \underline{u}(t; \alpha)) + i f(t, \overline{x}(t; \alpha), \overline{u}(t; \alpha)); \\ \underline{x}(a; \alpha) + i\overline{x}(a; \alpha) = (q\alpha + p(1 - \alpha)) + i(q\alpha + r(1 - \alpha)); \end{cases}$$

Based on concept of complex number, the new description of the problem (17) can be turned in to two problems (18) and (19):

$$\text{Min: } \int_a^b f_0(t, \underline{x}(t; \alpha), \underline{u}(t; \alpha))\, dt \qquad (18)$$

$$\text{S. to:} \begin{cases} (D_{a+}^{\beta} \underline{x})(t; \alpha) = f(t, \underline{x}(t; \alpha), \underline{u}(t; \alpha)); \\ \underline{x}(a; \alpha) = (q\alpha + p(1 - \alpha)), \end{cases}$$

and

$$\text{Min: } \int_a^b f_0(t, \overline{x}(t; \alpha), \overline{u}(t; \alpha))dt \qquad (19)$$

$$\text{S. to: } \begin{cases} (D_{a+}^{\beta} \overline{x})(t; \alpha) = f(t, \overline{x}(t; \alpha), \overline{u}(t; \alpha)); \\ \overline{x}(a; \alpha) = (q\alpha + r(1 - \alpha)). \end{cases}$$

By using these two problems and solving them, for any given α we generate the optimal pairs $(\underline{x}^*(t; \alpha), \underline{u}^*(t; \alpha))$ and $(\overline{x}^*(t; \alpha), \overline{u}^*(t; \alpha))$ for (18) and (19) respectively; therefore, a solution for (16) can be produced in α-cut form as follows:

$$\tilde{x}^*(t, \alpha) = [\underline{x}^*(t; \alpha), \overline{x}^*(t; \alpha)]; \qquad \tilde{u}^*(t, \alpha) = [\underline{u}^*(t; \alpha), \overline{u}^*(t; \alpha)].$$

Note 4.1: Consider the Cauchy type problem:

$$\begin{cases} (D_{a+}^{\beta} y)(x) = f(x, y(x)), \; 0 < \beta \le 1 \\ (I_{a+}^{1-\beta} y)(a) = b, \qquad b \in R, \end{cases} \qquad (20)$$

that $f(x, y(x))$ is a real-valued continuous function in domain $G \subset R \times R$ such that $sup_{(x,y)\in G}|f(x, y)| \le \infty$ and it satisfies the Lipschitz condition. The solution of fractional system (20) is given in [2] as:

$$y(x) = \frac{b(x - a)^{\beta-1}}{\Gamma(\beta)} + \frac{1}{\Gamma(\beta)} \int_a^x \frac{f(t, y(t))dt}{(x - t)^{1-\beta}}, \; x > a, \; 0 < \beta \le 1. \qquad (21)$$

Also, consider the following Cauchy type problem for linear differential equation:

$$\begin{cases}(D_{a+}^{\beta}y)(x)-\lambda y(x)=f(x);\\ y(a)=b;\end{cases} \tag{22}$$

then the solution of this system is:

$$y(x)=bx^{\beta-1}E_{\beta,\beta}(\lambda(x-a)^{\beta})+\int_a^x(x-t)^{\beta-1}E_{\beta,\beta}(\lambda(x-t)^{\beta})f(t)dt \tag{23}$$

where $E_{\beta,\beta}$ is Mittag-Leffler function, that in general form it is defined by [2]:

$$E_{\alpha,\beta}(z):=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\ z\in\mathbb{C}; \tag{24}$$

Theorem 4.2: Let f(x) and g(x) be analytic on [a, b]. Then

$$D_{a^+}^{\alpha}(fg)=\sum_{k=0}^{\infty}\binom{\alpha}{k}(D_{a^+}^{\alpha-k}f)g^{(k)},\qquad \alpha\in\mathbb{R}^1$$

where $\binom{\alpha}{k}=\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\Gamma(k+1)}$.

Proof: see [2].

Riccati differential equation for fractional optimal control

In this section, we are going to present a new method for solving a fractional optimal control problem by applying a Riccati differential equation based on fractional minimal principle of Pontryagin [17, 18]. To this end, consider the fractional optimal control as following:

$$min\,J=\frac{1}{2}S(t_1)x^2(t_1)+\frac{1}{2}\int_{t_0}^{t_1}\{P(t).x^2(t)+2q(t).x(t).u(t)+r(t).u^2(t)\}dt$$

$$\text{S. to:} \tag{25}$$

$$(D_{t_0}^{\beta}x)(t)=a(t)x(t)+b(t)u(t);$$
$$x(t_0)=x_0.$$

The Hamiltonian of (25) can be shown:

$$H=\frac{1}{2}P(t)x^2+q(t).x(t).u(t)+\frac{1}{2}r(t).u^2(t)+\lambda(a(t)x(t)+b(t)u(t)); \tag{26}$$

Then, based on the fractional Pontryagin system in (15_B), we have:

$$(D_{t_0}^{\beta}\lambda)(t)=\frac{\partial H}{\partial x}=px+qu+\lambda a \tag{27}$$

$$\frac{\partial H}{\partial u}=qx+ru+\lambda a=0 \tag{28}$$

(28) gives us:

$$u=-r^{-1}(qx-\lambda b) \tag{29}$$

Now, by applying (29) in the differential equation of problem (25), we have:

$$D_{t_0}^{\beta}x=(a-r^{-1}q)x-r^{-1}\lambda b^2 \tag{30}$$

In the similar way, we obtain:

$$D_{t_1}^{\beta}\lambda=(p-r^{-1}q^2)x+(-r^{-1}qb+a)\lambda \tag{31}$$

Now, let $\lambda(t_1)=S(t_1)x(t_1)$. Since the above system is linear, we can display the solution of it by final solution; it means that we can show it as following:

$$\begin{pmatrix}x(t)\\ \lambda(t)\end{pmatrix}=\phi(t,t_1)\begin{pmatrix}x(t_1)\\ \lambda(t_1)\end{pmatrix} \tag{32}$$

where ϕ is a matrix in dimension 2×2 and dependent to t; now, assume that:

$$\phi(t,t_1)=\begin{pmatrix}F(t,t_1) & G(t,t_1)\\ L(t,t_1) & M(t,t_1)\end{pmatrix}; \tag{33}$$

therefor, from (32) we have:

$$x(\mathrm{t})=\mathrm{F}x(t_1)+G\lambda(t_1) \tag{34}$$
$$\lambda(t)=\mathrm{L}x(t_1)+M\lambda(t_1) \tag{35}$$

Now by use the assumption $\lambda(t_1)=s(t_1)x(t_1)$ we obtain:

$$x(\mathrm{t})=(\mathrm{F}-\mathrm{GS})x(t_1); \tag{36}$$
$$\lambda(t)=(\mathrm{L}-\mathrm{MS})x(t_1) \tag{37}$$

By assuming $det(\mathrm{F}-\mathrm{GS})\neq 0$, from (36) and (37) we have:

$$\lambda(\mathrm{t})=(\mathrm{L}-\mathrm{MS})(\mathrm{F}-\mathrm{GS})^{-1}x(t). \tag{38}$$

Now, let

$$k(t)=(\mathrm{L}-\mathrm{MS})(\mathrm{F}-\mathrm{GS})^{-1}; \tag{39}$$

so

$$\lambda(\mathrm{t})=k(t)x(t) \tag{40}$$

According to Theorem (4.2) for $0<\beta<1$, we have:

$$(\mathrm{D}_{t_0+}^{\beta}\lambda)(\mathrm{t})=\binom{\beta}{0}(\mathrm{D}_{t_0+}^{\beta}\mathrm{k})(\mathrm{t}).x(\mathrm{t})=(\mathrm{D}_{t_0+}^{\beta}\mathrm{k})(\mathrm{t}).x(\mathrm{t}) \tag{41}$$

By substituting (31) and (32) in (41) we reach to the following equation:

$$(p - r^{-1}. q^2). x(t) + (-r^{-1}. q. b + a). \lambda(t) = (\mathrm{D}^{\beta}_{t_0^+}\mathrm{k})(\mathrm{t}). x(\mathrm{t}); \qquad (42)$$

then replacing λ in (42) by (40) gives:

$$(p - r^{-1}. q^2). x(t) + (-r^{-1}. q. b + a). k(t). x(t) = (\mathrm{D}^{\beta}_{t_0^+}\mathrm{k})(\mathrm{t}). x(\mathrm{t})$$

Because we are working for a nontrivial solution ($x \neq 0$), we obtain the Riccati differential equation to solve:

$$(p - r^{-1}. q^2) + (-r^{-1}. q. b + a). k(t) = (\mathrm{D}^{\beta}_{t_0^+}\mathrm{k})(\mathrm{t}). \qquad (43)$$

Now, regarding the formula (32), in $t = t_1$ we must have $= I_{2\times 2}$; so we must have $F = M = 1$ and $L = G = 0$. Hence, by using (39) the initial condition $k(t_1) = s(t_1)$ is obtained. Therefore, the function $k(t)$ can be determined by solving the following differential system that was discussed in note 4.1:

$$\begin{cases} (\mathrm{D}^{\beta}_{t_0^+}k)(\mathrm{t}) = (p - r^{-1}. q^2) + (-r^{-1}. q. b + a). k(t) \\ k(t_1) = s(t_1) \end{cases} \qquad (44)$$

By calculating $k(t)$ from (44) and substituting it in the (40), one can determine the $\lambda(t)$ according to $x(t)$. Next, this fact make (30) a fractional differential equation with an initial condition; By solving it we can compute the optimal trajectory for problem (25); then, the formula (29) give us the optimal control of it.

Numerical Example

Consider the fractional-fuzzy optimal control problem:

$$\text{Min: } \frac{1}{2}\tilde{x}(t_1)^2 + \frac{1}{2}\int_0^1 (\tilde{x}(t)^2 + 4\tilde{x}(t).\tilde{u}(t) + \tilde{u}(t)^2)dt$$

$$\text{S. to: } \begin{cases} (D^{\beta}_{0^+}\tilde{x})(t) = \tilde{x}(t) + \tilde{u}(t); \\ \tilde{x}(0) = (0,1,2), \end{cases} \qquad (45)$$

where $\tilde{x}(t)$ is fuzzy trajectory variable, $\tilde{u}(t)$ is fuzzy control variable, $t \in [0,1]$ and $0 < \beta \leq 1$. Based on (17), first we demonstrate the problem in complex space by using α-cut concept. According to the problems (18) and (19), we divide problem (45) in to the following sub-problems:

$$\text{Min: } \frac{1}{2}\underline{x}(t_1; \alpha)^2 + \frac{1}{2}\int_0^1 (\underline{x}(t; \alpha)^2 + 4\underline{x}(t; \alpha). \underline{u}(t; \alpha) + \underline{u}(t; \alpha)^2)dt$$

$$\text{S. to: } \begin{cases} (D^{\beta}_{0^+}\underline{x})(t; \alpha) = \underline{x}(t; \alpha) + \underline{u}(t; \alpha); \\ \underline{x}(0; \alpha) = \alpha. \end{cases} \qquad (46)$$

and

$$\text{Min: } \frac{1}{2}\overline{x}(t_1; \alpha)^2 + \frac{1}{2}\int_0^1 (\overline{x}(t; \alpha)^2 + 4\overline{x}(t; \alpha). \overline{u}(t; \alpha) + \overline{u}(t; \alpha)^2)dt$$

$$\text{S. to: } \begin{cases} (D^{\beta}_{0^+}\overline{x})(t; \alpha) = \overline{x}(t; \alpha) + \overline{u}(t; \alpha); \\ \overline{x}(0; \alpha) = \alpha + 2(1 - \alpha). \end{cases} \qquad (47)$$

First, for testing the method, let $\alpha = \beta = 1$. Then we have the optimal control problem as below and we solve it by Riccati differential equation that is presented by Pinch in [17] for optimal control problem with ordinary differential equation:

$$\text{Min: } \frac{1}{2} x(t_1)^2 + \frac{1}{2}\int_0^1 (x(t)^2 + 4u(t). x(t) + u(t)^2)dt$$

$$\text{S. to: } \begin{cases} \dot{x}(t) = x(t) + u(t); \\ x(0) = 1. \end{cases} \qquad (48)$$

The optimal trajectory and optimal control are shown in Figure 1:

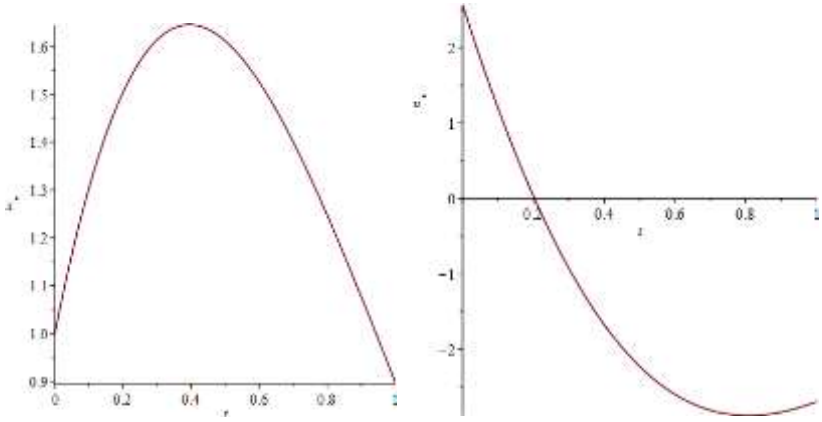

Figure 1: The optimal trajectory and optimal control for problem (45) where $\alpha = \beta = 1$

Now, we try to solve (46) by the new method presented in this paper; by applying the method, we reach to (44) in the following form:

$$\begin{cases} ({}_{t_0^+}D^{\beta} k)(t) = -1 - k(t) \\ k(1) = 1 \end{cases} \qquad (49)$$

Then, this fractional system can be solved by RUIM method [7] and the following solution is obtained:

$$k(t) = -2 + 2E_{\beta}(-t^{\beta}), \qquad (50)$$

that for $\beta = 1$ we have:

$$k(t) = -2 + 3e^{-t}. \qquad (51)$$

Therefore, from (40) we obtain:

$$\lambda(t) = (-2 + 2e^{-t})x(t). \qquad (52)$$

By substituting (52) in (30). We have the initial condition $x(0;\alpha) = \alpha$. One can obtain the optimal trajectory of this problem by solving the following fractional system:

$$\begin{cases} {}_{0^+}D^{\beta} \underline{x}(t;\alpha) = (1 - 2e^{-t}).\underline{x}(t;\alpha) \\ \underline{x}(0;\alpha) = \alpha. \end{cases} \qquad (53)$$

Similarly, we have the same fractional equation for $\overline{x}(t)$ which is the same as (53):

$$\begin{cases} {}_{0^+}D^{\beta} \overline{x}(t;\alpha) = (1 - 2e^{-t}).\overline{x}(t;\alpha) \\ \overline{x}(0;\alpha) = \alpha + 2(1-\alpha). \end{cases} \qquad (54)$$

For $\beta = \alpha = 1$, we have:

$x^*(t) = x(t) = \overline{x}(t) =$
$-(-(1/3) * exp(-3/exp(t))/(-1 + 3/exp(t)) + (1/6) * (exp(3) * exp(-3) - 3)/exp(3)) * (exp(3 * exp(-t) + t) - 3 * exp(3 * exp(-t))) + 2.$ (55)

The solutions of two methods are shown in the figure 2 for comparing with each other. The red curve is the optimal trajectory of problem (48) that is solved by Riccati differential equation method in [17] and the blue curve is the formula (55).

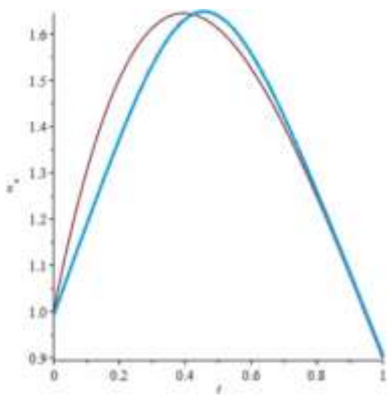

Figure 2. Comparing the Optimal trajectories by two methods for $\alpha = \beta = 1$

By using formula (25) and replacing $\lambda(t)$ from (52) in it, we have:

$$u^* = (-4 + 2e^{-t})x^*(t). \qquad (56)$$

Now, we apply $x^*(t)$ from (55) in (56) to determine the optimal control for $\alpha = \beta = 1$, that is shown in figure 3 by blue curve; and the red curve is the optimal control that is shown in figure 1.

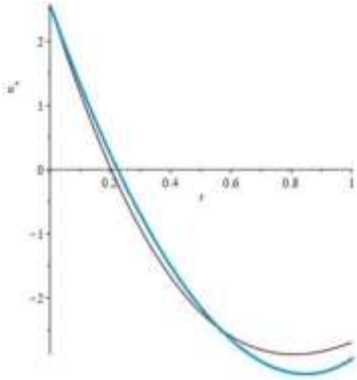

Figure 3. Comparing the optimal controls by two methods for $\alpha = \beta = 1$

Note that the original problem is free at end point. In this part of our example, we let $\beta = 0.5$ and obtain optimal trajectories for different values of α by (53) and (54); also, those are shown in figure 4.

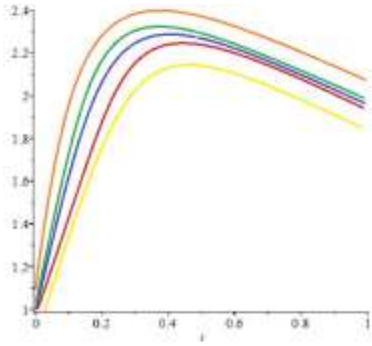

Figure 4. The optimal trajectories for $\beta = 0.5$ and different values of α

In the figure 4, the blue cure is optimal trajectory for $\alpha = 1$, the green and red curves are upper and lower optimal trajectory for $\alpha = 0.9$ and the orange and yellow curves are upper and lower optimal trajectory for $\alpha = 0.8$. As figure shown when the value of α comes close to 1, the upper and lower optimal trajectories curves come closer to each other; in $\alpha = 1$ the upper and lower curves of optimal trajectories overlap on each other which is adapted with the obtained theoretical solution in (55).

Conclusion. In this paper, we proposed a new analytical method for fractional fuzzy optimal control problem with fuzzy initial condition. Also based on α-level concepts in fuzzy mathematics, by applying minimal principle of pontriyagin, the method shows how the optimal solution is determined via a Riccati differential equation. Therefore, the method is able to characterize the solution analytically while the other methods look for approximation numerical one. One can easily obtain the optimal trajectory and optimal control by this method. In the test example, it is shown that when the new method is applied for $\alpha = \beta = 1$, the resulted optimal trajectory and control by the method is very near to exact solution. Also, when the values of α come close to one, the resulted curves come closer and closer to the exact solution, this fact indicates the convergence and exactness of the proposed method.

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POST-COLONIAL READING OF OTHELLO'S PLAY

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Abstract. On the horizon of literary criticism today, text just does not have a general meaning, but any text is in a state of waiting to read readers and reproduce new meanings. Shakespeare's texts are texts that are repeatedly presented with different types of readings from different perspectives, and each time a new reproduction of them is obtained. The present research seeks to present a new reading of Shakespeare's Othello drama using post-colonial studies. The Othello play has qualities that can be considered postcolonial. Therefore, the present paper, with a post-colonial approach, reciprocates Edward Said views with a qualitative, analytical, and descriptive approach to reading this play, referring to a large historical and political structure.

Keywords: Post-Colonial Studies-Edward Said-Othello-William Shakespeare.

Introduction. In the world of literature and art today, no longer has any meaning for any work of any kind. Hence, every text can be read by countless approaches. Each reader engages in a different way, depending on the approach it has adopted, and reproduces the meanings corresponding to that reading. Basically, the text opens the window to the reader to read another text. Shakespeare, with its diverse and multifaceted works, is one of the most important literary fields in the world. Shakespeare's texts, including plays or poems, are those works that have been repeatedly referred to by the contemporary era and have repeatedly been subject to various types of readings and studies. One of the most important of these texts is Othello, which has been repeatedly analyzed by feminism, semiotics, post-colonialism, and so on. Post-colonial studies dating back to the 80's have opened up a new window to cultural studies and literary readings. These studies deal with the collision of East and West and colonialism that has emerged. Edward Said, one of the most important post-colonial thinkers, still believes that colonialism has not been completed and continues in various ways. Hence, a series of post-colonial works to critique and re-read what has already been accumulated under the name of literature. The need for this re-reading is to pinpoint and highlight those lines that have come from the memory of colonialism into literature.

With the growth of post-colonial studies, there has been much research on Shakespeare's texts. Unfortunately, in the Persian language so far, there has not been much attention to the necessity of using these ideas in the critique of literature and theater. Post-colonial studies are so widespread and cover so vast a wide range of subjects as political,